

# Central Configurations Formed By Two Twisted Regular Polygons

XIANG YU\* AND SHIQING ZHANG†

*Department of Mathematics, Sichuan University, Chengdu 610064, China*

## Abstract

In this paper, we study the necessary conditions and sufficient conditions for the central configurations formed by two twisted regular polygons (one N-regular polygon and one L-regular polygon). We wish to extend the results of the symmetrical central configurations formed by two twisted N-regular polygons, however, it will be proved that there are not more central configurations in a more general setting than the central configurations considered for some more particular situations before.

**Keywords:** Twisted N+L-body problems, Central configurations.

**2000AMS Mathematical Subject Classification:** 70F10, 70F15.

## 1 Introduction

Central configuration plays a very important role in many problems, such as Newtonian N-body problems. It is highly concentrated by mathematicians [1, 2, 3, 4, 5, 6, 7, 8, 9]. To find concrete central configurations is very difficult, therefore we consider only some special situation, i.e., the central configurations formed by two twisted regular polygons, which are an extension of the results in [3, 9]. The motivation of this paper comes mainly from the results of [3, 10, 9].

**Definition 1.1.** A configuration  $q = (q_1, q_2, \dots, q_n) \in X \setminus \Delta$  is called a central configuration, if there exists a constant  $\lambda \in \mathbb{R}$  such that

$$\sum_{j=1, j \neq k}^n \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, \quad 1 \leq k \leq n. \quad (1)$$

The value of  $\lambda$  in (1) is uniquely determined by

$$\lambda = \frac{U(q)}{I(q)}, \quad (2)$$

where

$$X = \{q = (q_1, q_2, \dots, q_n) \in \mathbb{R}^{3n} \mid \sum_{i=1}^n m_i q_i = 0\}, \quad (3)$$

$$\Delta = \{q \mid q_j = q_k \text{ for some } j \neq k\}, \quad (4)$$

$$U(q) = \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{|q_j - q_k|}, \quad (5)$$

$$I(q) = \sum_{1 \leq j \leq n} m_j |q_j|^2. \quad (6)$$

---

\*Email: xiang.zhiy@gmail.com

†Email: zhangshiqing@msn.com

Consider the central configurations in  $\mathbb{R}^3$  formed by one regular N-polygon and another regular L-polygon with distance  $h \geq 0$  (without loss of generality, we set  $N \leq L$ ). It is assumed that the lower layer regular N-polygon lies in horizontal plane, and the upper regular L-polygon parallels to the lower one and z-axis passes through both centers of two regular polygons. Suppose that the lower layer particles have masses  $m_1, m_2, \dots, m_N$  and the upper layer particles have masses  $\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_L$  respectively. For convenience, we may treat  $\mathbf{R}^3$  as the direct product of the complex plane and real axis when choosing the coordinates. Let  $\rho_k$  be the  $k$ -th root of the N-roots of unity,  $\xi_l$  be the  $l$ -th root of the L-roots of unity, i.e.,

$$\rho_k = e^{i\theta_k}, \quad (7)$$

$$\xi_l = e^{i\varphi_l}, \quad (8)$$

and let

$$\tilde{\rho}_l = a\xi_l \cdot e^{i\theta}, \quad (9)$$

where  $a > 0, i = \sqrt{-1}, \theta_k = \frac{2k\pi}{N} (k = 1, 2, \dots, N), \varphi_l = \frac{2l\pi}{L} (l = 1, 2, \dots, L), 0 \leq \theta \leq 2\pi$ ,  $\theta$  is called twisted angle.

It is assumed that  $m_1, m_2, \dots, m_N$  locates at the vertex  $q_k$  of the lower regular N-polygon;  $\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_L$  locates at the vertex  $\tilde{q}_l$  of the upper regular L-polygon, where

$$q_k = (\rho_k, 0), \quad \tilde{q}_l = (\tilde{\rho}_l, h). \quad (10)$$

The center of mass is

$$z_0 = \frac{\sum_j m_j q_j + \sum_l \tilde{m}_l \tilde{q}_l}{M}, \quad (11)$$

where

$$M = \sum_j m_j + \sum_l \tilde{m}_l. \quad (12)$$

Let

$$P_k = q_k - z_0, \quad P = (P_1, P_2, \dots, P_N), \quad (13)$$

$$\tilde{P}_l = \tilde{q}_l - z_0, \quad \tilde{P} = (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L). \quad (14)$$

If  $P_1, P_2, \dots, P_N; \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L$  form a central configuration, then  $\exists \lambda \in \mathbb{R}^+$ , such that

$$\sum_{j=1, j \neq k}^N \frac{m_j}{|P_k - P_j|^3} (P_k - P_j) + \sum_{j=1}^L \frac{\tilde{m}_j}{|P_k - \tilde{P}_j|^3} (P_k - \tilde{P}_j) = \lambda P_k, 1 \leq k \leq N, \quad (15)$$

$$\sum_{j=1, j \neq k}^L \frac{\tilde{m}_j}{|\tilde{P}_l - \tilde{P}_j|^3} (\tilde{P}_l - \tilde{P}_j) + \sum_{j=1}^N \frac{m_j}{|\tilde{P}_l - P_j|^3} (\tilde{P}_l - P_j) = \lambda \tilde{P}_l, 1 \leq l \leq L. \quad (16)$$

In the following, we only consider the case of  $m_1 = \dots = m_N = m$  and  $\tilde{m}_1 = \dots = \tilde{m}_L = bm$ . Then

$$z_0 = \sum_j (m_j q_j + \tilde{m}_j \tilde{q}_j) / M = (0, 0, \frac{bLh}{N + bL}), \quad (17)$$

and the necessary conditions and sufficient conditions for  $P_1, P_2, \dots, P_N; \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L$  forming a central configuration are

$$A + b \sum_{j=1}^L \frac{1 - a \cos(\frac{2\pi j}{L} - \frac{2\pi k}{N} + \theta)}{[1 + a^2 - 2a \cos(\frac{2\pi j}{L} - \frac{2\pi k}{N} + \theta) + h^2]^{\frac{3}{2}}} = \mu, \quad (1 \leq k \leq N) \quad (18)$$

$$bB + \sum_{j=1}^N \frac{1 - a^{-1} \cos(\frac{2\pi j}{N} - \frac{2\pi k}{L} - \theta)}{[1 + a^2 - 2a \cos(\frac{2\pi j}{N} - \frac{2\pi k}{L} - \theta) + h^2]^{\frac{3}{2}}} = \mu, \quad (1 \leq k \leq L) \quad (19)$$

$$\sum_{j=1}^L \frac{\sin(\frac{2\pi j}{L} - \frac{2\pi k}{N} + \theta)}{[1 + a^2 - 2a \cos(\frac{2\pi j}{L} - \frac{2\pi k}{N} + \theta) + h^2]^{\frac{3}{2}}} = 0, \quad (1 \leq k \leq N) \quad (20)$$

$$\sum_{j=1}^N \frac{\sin(\frac{2\pi j}{N} - \frac{2\pi k}{L} - \theta)}{[1 + a^2 - 2a \cos(\frac{2\pi j}{N} - \frac{2\pi k}{L} - \theta) + h^2]^{\frac{3}{2}}} = 0, \quad (1 \leq k \leq L) \quad (21)$$

$$h \sum_{j=1}^L \frac{1}{[1 + a^2 - 2a \cos(\frac{2\pi j}{L} - \frac{2\pi k}{N} + \theta) + h^2]^{\frac{3}{2}}} = \frac{\mu L h}{N + bL}, \quad (1 \leq k \leq N) \quad (22)$$

$$h \sum_{j=1}^N \frac{1}{[1 + a^2 - 2a \cos(\frac{2\pi j}{N} - \frac{2\pi k}{L} - \theta) + h^2]^{\frac{3}{2}}} = \frac{\mu N h}{N + bL}, \quad (1 \leq k \leq L) \quad (23)$$

where

$$A = \sum_{j=1}^{N-1} \frac{1 - \rho_j}{|1 - \rho_j|^3} > 0, \quad B = \sum_{j=1}^{L-1} \frac{1 - \xi_j}{|1 - \xi_j|^3} > 0, \quad \mu = \frac{\lambda}{m}. \quad (24)$$

For the central configurations of this type, R. Moeckel and C. Simo [3] proved the following results with condition  $\theta = 0$ ,  $L = N$ :

**Theorem 1.2.** (*R. Moeckel and C.Simo*). When  $h = 0, \theta = 0$ , for every mass ratio  $b$ , there are exactly two planar central configurations consisting of two nested regular  $N$ -polygon. For one of these, the ratio  $a$  of the sizes of the two polygons is less than 1, and for the other it is greater than 1. However, for  $N \geq 473$  there is a constant  $b_0(N) < 1$  such that for  $b < b_0$  and  $b > \frac{1}{b_0}$ , the central configuration with the smaller masses on the inner polygon is a repeller.

**Theorem 1.3.** (*R. Moeckel and C.Simo*). When  $h^2 > 0, \theta = 0$ , if  $N < 473$ , there is a unique pair of spatial central configurations of parallel regular  $N$ -polygon. If  $N \geq 473$ , there are no such central configurations for  $b < b_0(N)$ . At  $b = b_0$  a unique pair bifurcates from the planar central configuration with the smaller masses on the inner polygon. This remains the unique pair of spatial central configurations until  $b = \frac{1}{b_0}$ , where a similar bifurcation occurs in reverse, so that for  $b > \frac{1}{b_0}$ , only the planar central configurations remain.

X. Yu and S. Q. Zhang [9] proved the following results with condition  $L = N$ :

**Theorem 1.4.** If the central configuration is formed by two twisted regular  $N$ -polygon ( $N \geq 2$ ) with distance  $h \geq 0$ , then only  $\theta = 0$  or  $\theta = \pi/N$ . Specifically, if  $a = 1$  and  $h = 0$ , i.e., two nested regular  $N$ -polygon are on the same unit circle, then only  $\theta = \pi/N$ .

**Corollary 1.5.** For  $N \geq 2, h = 0$ , if  $a = 1$ , then  $b = 1$  and  $\theta = \pi/N$ , i.e., there is exactly one central configuration formed by two nested regular  $N$ -polygon on the same unit circle, which is the regular  $2N$ -polygon.

**Corollary 1.6.** The configuration formed by two twisted regular  $N$ -polygon ( $N \geq 2$ ) with distance  $h \geq 0$  is a central configuration if and only if the parameters  $a, b, h$  satisfy the following relationships: i. When  $h = 0$  and  $a \neq 1$

$$b \left[ \sum_{1 \leq j \leq N} \frac{1 - a \cos(\theta_j)}{(1 + a^2 - 2a \cos(\theta_j))^{3/2}} - \frac{A}{a^3} \right] = \sum_{1 \leq j \leq N} \frac{1 - a^{-1} \cos(\theta_j)}{(1 + a^2 - 2a \cos(\theta_j))^{3/2}} - A \quad (25)$$

or

$$b \left[ \sum_{1 \leq j \leq N} \frac{1 - a \cos(\theta_j + \frac{\pi}{N})}{(1 + a^2 - 2a \cos(\theta_j + \frac{\pi}{N}))^{3/2}} - \frac{A}{a^3} \right] = \sum_{1 \leq j \leq N} \frac{1 - a^{-1} \cos(\theta_j + \frac{\pi}{N})}{(1 + a^2 - 2a \cos(\theta_j + \frac{\pi}{N}))^{3/2}} - A. \quad (26)$$

ii. When  $h > 0$

$$\begin{cases} ba \sum_{1 \leq j \leq N} \frac{\cos(\theta_j)}{(1+a^2-2a \cos(\theta_j)+h^2)^{3/2}} = A - \sum_{1 \leq j \leq N} \frac{1}{(1+a^2-2a \cos(\theta_j)+h^2)^{3/2}} \\ ba \left( \frac{A}{a^3} - \sum_{1 \leq j \leq N} \frac{1}{(1+a^2-2a \cos(\theta_j)+h^2)^{3/2}} \right) = \sum_{1 \leq j \leq N} \frac{\cos(\theta_j)}{(1+a^2-2a \cos(\theta_j)+h^2)^{3/2}} \end{cases} \quad (27)$$

or

$$\begin{cases} ba \sum_{1 \leq j \leq N} \frac{\cos(\theta_j + \frac{\pi}{N})}{(1+a^2-2a \cos(\theta_j + \frac{\pi}{N})+h^2)^{3/2}} = A - \sum_{1 \leq j \leq N} \frac{1}{(1+a^2-2a \cos(\theta_j + \frac{\pi}{N})+h^2)^{3/2}} \\ ba \left( \frac{A}{a^3} - \sum_{1 \leq j \leq N} \frac{1}{(1+a^2-2a \cos(\theta_j + \frac{\pi}{N})+h^2)^{3/2}} \right) = \sum_{1 \leq j \leq N} \frac{\cos(\theta_j + \frac{\pi}{N})}{(1+a^2-2a \cos(\theta_j + \frac{\pi}{N})+h^2)^{3/2}}. \end{cases} \quad (28)$$

**Corollary 1.7.** For  $N \geq 2, h > 0, a = 1$ , if the configuration formed by two twisted regular  $N$ -polygon ( $N \geq 2$ ) with distance  $h \geq 0$  is a central configuration, then  $b = 1, \theta = 0$  or  $\pi/N$ , and there exists a unique  $h$  for each  $\theta$ . In other words, there are exactly two spatial central configurations formed by parallel regular  $N$ -polygon which have the same sizes.

In this paper, we will prove the following main result:

**Theorem 1.8.** If  $P_1, P_2, \dots, P_N; \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L$  form a central configuration, then  $N = L$ , i.e., two stacked regular polygons forming a symmetrical central configuration (considered by us) have the same shape.

Then the necessary conditions and sufficient conditions for the central configurations formed by two twisted regular polygons are the conditions given in Corollary 1.5 and 1.6. Furthermore, some particular cases are already completely known, specially, the case of  $\theta = 0$  in Theorem 1.2 and 1.3.

## 2 Some Lemmas

**Lemma 2.1.** Let  $a_j > 0, 1 \leq j \leq k, A_1 \geq A_2 \geq \dots \geq A_k \geq 0$ , then

$$\lim_{n \rightarrow \infty} \left( \sum_{1 \leq j \leq k} a_j A_j^n \right)^{\frac{1}{n}} = A_1. \quad (29)$$

**Lemma 2.2.** Given  $a_j > 0, 1 \leq j \leq k, A_1 > \dots > A_k > 0$ . For the function  $f(x) = (\sum_{1 \leq j \leq k} a_j A_j^x)^{\frac{1}{x}}, x \in (0, \infty)$ , we have

$$f'(x) = (-A_1 \ln a_1) \frac{1}{x^2} + o\left(\frac{1}{x^2}\right), \quad (30)$$

when  $x \rightarrow \infty$ .

*Proof.*  $f'(x) = (\sum_{1 \leq j \leq k} a_j A_j^x)^{\frac{1}{x}} \left[ \frac{x \sum_{1 \leq j \leq k} a_j A_j^x \ln A_j}{\sum_{1 \leq j \leq k} a_j A_j^x} - \ln \sum_{1 \leq j \leq k} a_j A_j^x \right] / x^2$   
 $= \frac{1}{x^2} (\sum_{1 \leq j \leq k} a_j A_j^x)^{\frac{1}{x}} \left[ \frac{\ln A_1 + \sum_{2 \leq j \leq k} \frac{b_j B_j^x \ln A_j}{1 + \sum_{2 \leq j \leq k} b_j B_j^x} x - \ln(a_1 A_1^x) - \ln 1 + \sum_{2 \leq j \leq k} b_j B_j^x \right].$

where  $b_j = \frac{a_j}{a_1}, B_j = \frac{A_j}{A_1} (2 \leq j \leq k)$ .

Then  $B_j^x \rightarrow 0, x B_j^x \rightarrow 0$ , when  $x \rightarrow \infty$ .

So  $f'(x) = \frac{1}{x^2} (\sum_{1 \leq j \leq k} a_j A_j^x)^{\frac{1}{x}} [-\ln a_1 + o(1)] = (-A_1 \ln a_1) \frac{1}{x^2} + o\left(\frac{1}{x^2}\right).$   $\square$

**Lemma 2.3.** Let

$$g(x, \alpha) = \sum_{1 \leq j \leq N} \frac{\sin(\theta_j + \theta)}{(1 + a^2 - 2a \cos(\theta_j + \theta) + x)^\alpha},$$

where  $\theta \in (0, \frac{\pi}{N})$ ,  $a > 0, \alpha > 0, x \geq 0$ . Then  $g(x, \alpha) > 0$  in  $\{x : x \geq 0\}$  provided  $\alpha$  is sufficiently large.

*Proof.* Set  $t = \frac{2a}{1+a^2+x}$ , then  $t \in (0, \frac{2a}{1+a^2}]$  and we need only to prove that

$$f(t, \alpha) = \sum_{1 \leq j \leq N} \frac{\sin(\theta_j + \theta)}{(1 - t \cos(\theta_j + \theta))^\alpha}$$

is positive in  $(0, \frac{2a}{1+a^2}]$  for sufficiently large  $\alpha$ .

Firstly, we have

$$\begin{aligned} f(t, \alpha) &= \sum_{1 \leq j \leq N} \sin(\theta_j + \theta) \sum_{m \geq 0} \left[ \frac{\alpha(\alpha+1) \cdots (\alpha+m-1)}{m!} t^m \cos^m(\theta_j + \theta) \right] \\ &= \sum_{m \geq 0} \left[ \frac{\alpha(\alpha+1) \cdots (\alpha+m-1)}{m!} \left(\frac{t}{2}\right)^m \sum_{1 \leq j \leq N} [\sin(m+1)(\theta_j + \theta) + (m-1)\sin(m-1)(\theta_j + \theta) + \cdots] \right] \\ &= \frac{\alpha(\alpha+1) \cdots (\alpha+N-2)}{(N-1)!} \left(\frac{t}{2}\right)^{N-1} N \sin(N\theta) + o(t^{N-1}) \end{aligned}$$

So there exists  $\delta_\alpha$  for any given  $\alpha$  such that  $f(t, \alpha)$  is positive for  $t \in (0, \delta_\alpha]$ .

Let

$$h(t, \alpha) = \left\{ \sum_{0 \leq j \leq [\frac{N-1}{2}]} \frac{\sin(\theta_j + \theta)}{(1 - t \cos(\theta_j + \theta))^\alpha} \right\}^{\frac{1}{\alpha}} - \left\{ \sum_{1 \leq j \leq [\frac{N}{2}]} \frac{\sin(\theta_j - \theta)}{(1 - t \cos(\theta_j - \theta))^\alpha} \right\}^{\frac{1}{\alpha}},$$

then  $h(t, \alpha)$  and  $f(t, \alpha)$  have the same sign.

We have  $\frac{\partial h(t, \alpha)}{\partial \alpha} = \left[ -\frac{\ln(\sin \theta)}{1 - t \cos \theta} + \frac{\ln(\sin(\frac{2\pi}{N} - \theta))}{1 - t \cos(\frac{2\pi}{N} - \theta)} \right] \frac{1}{\alpha^2} + o(\frac{1}{\alpha^2})$ , since  $\frac{\ln(\sin(\frac{2\pi}{N} - \theta))}{1 - t \cos(\frac{2\pi}{N} - \theta)} - \frac{\ln(\sin \theta)}{1 - t \cos \theta} = \frac{(1 - t \cos \theta)[\ln(\sin(\frac{2\pi}{N} - \theta)) - \ln(\sin \theta)] + t[\cos(\frac{2\pi}{N} - \theta) - \cos \theta]}{[1 - t \cos(\frac{2\pi}{N} - \theta)](1 - t \cos \theta)} > 0$ , hence  $h(t, \alpha)$  is increasing about  $\alpha$  provided  $\alpha$  is sufficiently large.

So there exists some positive number  $\delta$  (independent of  $\alpha$ ) such that  $h(t, \alpha)$  is positive in  $(0, \delta]$  for sufficiently large  $\alpha$ .

Secondly,  $h(t, \alpha) \rightarrow h(t)$  for any  $t \in [\delta, \frac{2a}{1+a^2}]$ , where  $h(t) = \frac{1}{1 - t \cos \theta} - \frac{1}{1 - t \cos(\frac{2\pi}{N} - \theta)}$ , and there exists some positive number  $\epsilon$  such that  $h(t) \geq \epsilon > 0$  for any  $t \in [\delta, \frac{2a}{1+a^2}]$ .

Since  $h(t, \alpha)$  is increasing about  $\alpha$  provided  $\alpha$  is sufficiently large, by the well known theorem (Dini), we know that  $h(t, \alpha) \rightrightarrows h(t)$  on the compact interval  $[\delta, \frac{2a}{1+a^2}]$ , thus  $h(t, \alpha)$  is positive in  $[\delta, \frac{2a}{1+a^2}]$  for sufficiently large  $\alpha$ .

As a result  $g(x, \alpha)$  is positive in  $\{x : x \geq 0\}$  provided  $\alpha$  is sufficiently large.  $\square$

**Remark.** We say some words about the proof of the **Lemma 2.3** here. We found there was a gap in our original proof of the important **Lemma 2.10** in [9]. Since we didn't notice that the problem of the uniform convergence of  $h(t, \alpha)$  and the compactness of the whole interval we considered. So here we give the new proof of the **Lemma 2.3** to correct the error in **Lemma 2.10** of [9].

Then we have the following important proposition which is a corollary of **Lemma 2.3**. The detailed proof can be found in [9].

**Lemma 2.4.** *If*

$$\sum_{j=1}^N \frac{\sin(\theta_j + \theta)}{[1 + a^2 - 2a \cos(\theta_j + \theta) + h^2]^{\frac{3}{2}}} = 0, \quad (31)$$

then  $\theta = \frac{j\pi}{N} \pmod{2\pi}$  for some  $1 \leq j \leq 2N$ .

### 3 The proof of main results

By(20) and **lemma 2.4**, for  $\forall k \in \{1, 2, \dots, L\}$  there exist corresponding  $l_k$  such that  $1 \leq l_k \leq 2L$  and

$$\theta - \frac{2\pi k}{N} = \frac{l_k \pi}{L} (\text{mod } 2\pi). \quad (32)$$

Then there exists some  $j \in \{1, 2, \dots, 2L\}$  such that  $\frac{2\pi}{N} = \frac{j\pi}{L} (\text{mod } 2\pi)$ , moreover we have  $N|(2L)$ . Similarly, by(21), for  $\forall \nu \in \{1, 2, \dots, N\}$  there exist corresponding  $n_\nu$  such that  $1 \leq n_\nu \leq 2N$  and

$$-\theta - \frac{2\pi \nu}{L} = \frac{n_\nu \pi}{N} (\text{mod } 2\pi). \quad (33)$$

Then there exists some  $j \in \{1, 2, \dots, 2N\}$  such that  $\frac{2\pi}{L} = \frac{j\pi}{N} (\text{mod } 2\pi)$ , thus we also have  $L|(2N)$ . Let  $N = 2^n \cdot N_1, L = 2^l \cdot L_1$ , where  $N_1, L_1$  are odd, then we have  $N_1 = L_1$ , and  $n \leq l \leq n+1$ , so  $L = N$  or  $L = 2N$ .

In the following, we will prove  $L \neq 2N$ . Otherwise if  $L = 2N$ , we have  $\theta = \frac{(4k+l_k)\pi}{2N} (\text{mod } 2\pi)$  ( $1 \leq k \leq 2N, 1 \leq l_k \leq 4N$ ) by (32) and  $\theta = -\frac{(\nu+n_\nu)\pi}{N} (\text{mod } 2\pi)$  ( $1 \leq \nu \leq N, 1 \leq n_\nu \leq 2N$ ) by (33), hence  $l_k = 2l'_k$  ( $1 \leq l'_k \leq 2N$ ) for any  $k \in \{1, 2, \dots, 2N\}$  and  $2k + l'_k + \nu + n_\nu = 0 (\text{mod } 2N)$  for any  $k \in \{1, 2, \dots, 2N\}, \nu \in \{1, 2, \dots, N\}$ , furthermore, there are both even and odd number in the numbers  $n_\nu$  ( $\nu \in \{1, 2, \dots, N\}$ ).

Then it's easy to know that  $P_1, P_2, \dots, P_N; \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L$  form a central configuration if and only if  $a, b, h$  satisfy the following equations (in fact, by symmetry, one can get the same result):

$$A + b \sum_{j=1}^{2N} \frac{1 - a \cos(\frac{j\pi}{N})}{[1 + a^2 - 2a \cos(\frac{j\pi}{N}) + h^2]^{\frac{3}{2}}} = \mu, \quad (34)$$

$$bB + \sum_{j=1}^N \frac{1 - a^{-1} \cos(\frac{2j-1}{N}\pi)}{[1 + a^2 - 2a \cos(\frac{2j-1}{N}\pi) + h^2]^{\frac{3}{2}}} = \mu, \quad (35)$$

$$bB + \sum_{j=1}^N \frac{1 - a^{-1} \cos(\frac{2j}{N}\pi)}{[1 + a^2 - 2a \cos(\frac{2j}{N}\pi) + h^2]^{\frac{3}{2}}} = \mu, \quad (36)$$

$$h \sum_{j=1}^{2N} \frac{1}{[1 + a^2 - 2a \cos(\frac{j\pi}{N}) + h^2]^{\frac{3}{2}}} = \frac{\mu L h}{N + bL}, \quad (37)$$

$$h \sum_{j=1}^N \frac{1}{[1 + a^2 - 2a \cos(\frac{2j-1}{N}\pi) + h^2]^{\frac{3}{2}}} = \frac{\mu N h}{N + bL}, \quad (38)$$

$$h \sum_{j=1}^N \frac{1}{[1 + a^2 - 2a \cos(\frac{2j}{N}\pi) + h^2]^{\frac{3}{2}}} = \frac{\mu N h}{N + bL}. \quad (39)$$

**Lemma 3.1.** *Let*

$$f(x) = \sum_{j=1}^N \frac{1}{[1 + a^2 - 2a \cos(\frac{2j-1}{N}\pi) + x]^{\frac{3}{2}}} - \sum_{j=1}^N \frac{1}{[1 + a^2 - 2a \cos(\frac{2j}{N}\pi) + x]^{\frac{3}{2}}} \quad x \in [0, +\infty), \quad (40)$$

*then for any  $a \in (0, \infty)$ , we have  $f(x) < 0$  for any  $x \in [0, +\infty)$  except the unique singularity  $a = 1, x = 0$ .*

*Proof.* Set  $t = \frac{2a}{1+a^2+x}$ , then  $t \in (0, 1)$  and we need only to prove that

$$g(t) = \sum_{1 \leq j \leq N} \frac{1}{(1 - t \cos(\theta_j - \frac{\pi}{N}))^{\frac{3}{2}}} - \sum_{1 \leq j \leq N} \frac{1}{(1 - t \cos \theta_j)^{\frac{3}{2}}}$$

is negative in  $(0, 1)$ .

In fact, we have

$$\begin{aligned}
g(t) &= \sum_{1 \leq j \leq N} \sum_{m \geq 0} \left[ \frac{(\frac{3}{2})(\frac{3}{2}+1) \cdots (\frac{3}{2}+m-1)}{m!} t^m (\cos^m(\theta_j - \frac{\pi}{N}) - \cos^m \theta_j) \right] \\
&= \sum_{m \geq 0} \left[ \frac{(\frac{3}{2})(\frac{3}{2}+1) \cdots (\frac{3}{2}+m-1)}{m!} \left( \frac{t^m}{2^{m-1}} \right) \right] \sum_{1 \leq j \leq N} \{ [\cos m(\theta_j - \frac{\pi}{N}) + m \cos(m-2)(\theta_j - \frac{\pi}{N}) + \cdots] \\
&\quad - [\cos m(\theta_j) + m \cos(m-2)(\theta_j) + \cdots] \}
\end{aligned}$$

Let  $a_m = [\cos m(\theta_j - \frac{\pi}{N}) + m \cos(m-2)(\theta_j - \frac{\pi}{N}) + \cdots] - [\cos m(\theta_j) + m \cos(m-2)(\theta_j) + \cdots]$ , then it's easy to know that  $a_m \leq 0$  and  $g(t)$  is negative in  $(0, 1)$ .  $\square$

By (38),(39) and **Lemma 3.1**, we know  $h = 0$ , and then there must be  $a \neq 1$ , otherwise there will be collision. So  $P_1, P_2, \dots, P_N; \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L$  form a central configuration if and only if  $a, b, h$  satisfy the following equations

$$A + b \sum_{j=1}^{2N} \frac{1 - a \cos(\frac{j\pi}{N})}{[1 + a^2 - 2a \cos(\frac{j\pi}{N})]^{\frac{3}{2}}} = \mu, \quad (41)$$

$$bB + \sum_{j=1}^N \frac{1 - a^{-1} \cos(\frac{2j-1}{N}\pi)}{[1 + a^2 - 2a \cos(\frac{2j-1}{N}\pi)]^{\frac{3}{2}}} = \mu, \quad (42)$$

$$bB + \sum_{j=1}^N \frac{1 - a^{-1} \cos(\frac{2j}{N}\pi)}{[1 + a^2 - 2a \cos(\frac{2j}{N}\pi)]^{\frac{3}{2}}} = \mu, \quad (43)$$

**Lemma 3.2.** *Let*

$$f(x) = \sum_{j=1}^N \frac{1 - x^{-1} \cos(\frac{2j-1}{N}\pi)}{[1 + x^2 - 2x \cos(\frac{2j-1}{N}\pi)]^{\frac{3}{2}}} - \sum_{j=1}^N \frac{1 - x^{-1} \cos(\frac{2j}{N}\pi)}{[1 + x^2 - 2x \cos(\frac{2j}{N}\pi)]^{\frac{3}{2}}}, x \in (0, 1) \cup (1, \infty). \quad (44)$$

then  $f(x) < 0$  for  $x \in (1, \infty)$  and  $f(x) > 0$  for  $x \in (0, 1)$ .

*Proof.* Let

$$h(x) = \sum_{j=1}^N \frac{1}{[1 + x^2 - 2x \cos(\frac{2j-1}{N}\pi)]^{\frac{1}{2}}} - \sum_{j=1}^N \frac{1}{[1 + x^2 - 2x \cos(\frac{2j}{N}\pi)]^{\frac{1}{2}}},$$

then

$$\frac{dh(x)}{dx} = -xf(x) \quad (45)$$

Firstly, we will prove  $f(x)$  is negative in  $(1, \infty)$ .

Set  $t(x) = \frac{2x}{1+x^2}$ , then  $t \in (0, 1)$  and  $\frac{dt(x)}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}$ ,  $\frac{dt(x)}{dx} > 0$  for  $x \in (0, 1)$ ,  $\frac{dt(x)}{dx} < 0$  for  $x \in (1, \infty)$ . Furthermore, we have  $h(x) = \frac{1}{(1+x^2)^{\frac{1}{2}}} g(t)$ , where

$$g(t) = \sum_{1 \leq j \leq N} \frac{1}{(1 - t \cos(\theta_j - \frac{\pi}{N}))^{\frac{1}{2}}} - \sum_{1 \leq j \leq N} \frac{1}{(1 - t \cos \theta_j)^{\frac{1}{2}}}$$

Since

$$\begin{aligned}
g(t) &= \sum_{1 \leq j \leq N} \sum_{m \geq 0} \left[ \frac{(\frac{1}{2})(\frac{1}{2}+1) \cdots (\frac{1}{2}+m-1)}{m!} t^m (\cos^m(\theta_j - \frac{\pi}{N}) - \cos^m \theta_j) \right] \\
&= \sum_{m \geq 0} \left[ \frac{(\frac{1}{2})(\frac{1}{2}+1) \cdots (\frac{1}{2}+m-1)}{m!} \left( \frac{t^m}{2^{m-1}} \right) \right] \sum_{1 \leq j \leq N} \{ [\cos m(\theta_j - \frac{\pi}{N}) + m \cos(m-2)(\theta_j - \frac{\pi}{N}) + \cdots] \\
&\quad - [\cos m(\theta_j) + m \cos(m-2)(\theta_j) + \cdots] \}
\end{aligned}$$

Let  $a_m = [\cos m(\theta_j - \frac{\pi}{N}) + m \cos(m-2)(\theta_j - \frac{\pi}{N}) + \dots] - [\cos m(\theta_j) + m \cos(m-2)(\theta_j) + \dots]$ , then it's easy to know that  $a_m \leq 0$  and the first nonzero term is  $a_N = -2$ , furthermore,  $g(t)$  and  $\frac{dg(t)}{dt}$  is negative in  $(0, 1)$ .

By (45), we have

$$\frac{x(1+x^2)^{\frac{5}{2}}f(x)}{2} = (1+x^2)g(t) + \frac{(x^2-1)}{x} \frac{dg(t)}{dt}, \quad (46)$$

hence  $f(x) < 0$  for  $x \in (1, \infty)$ .

Secondly, let us prove  $f(x)$  is positive in  $(0, 1)$ .

In the following, we will prove  $h(x)$  and  $\frac{dh(x)}{dx}$  are both negative in  $(0, 1)$  by the method of R. Moeckel and C. Simo [3].

Let

$$d(z) = \frac{1}{(1-z)^{\frac{1}{2}}} = \sum_{m \geq 0} d_m z^m,$$

then

$$\begin{aligned} h(x) &= \sum_{j=1}^N \frac{1}{[1 - x \exp(\frac{\sqrt{-1}(2j-1)}{N}\pi)]^{\frac{1}{2}}} \frac{1}{[1 - x \exp(\frac{-\sqrt{-1}(2j-1)}{N}\pi)]^{\frac{1}{2}}} \\ &- \sum_{j=1}^N \frac{1}{[1 - x \exp(\frac{\sqrt{-1}2j}{N}\pi)]^{\frac{1}{2}}} \frac{1}{[1 - x \exp(\frac{-\sqrt{-1}2j}{N}\pi)]^{\frac{1}{2}}} \\ &= \sum_{j=1}^N \sum_{m \geq 0} d_m x^m \exp(\frac{\sqrt{-1}m(2j-1)}{N}\pi) \sum_{m \geq 0} d_m x^m \exp(\frac{-\sqrt{-1}m(2j-1)}{N}\pi) \\ &- \sum_{j=1}^N \sum_{m \geq 0} d_m x^m \exp(\frac{\sqrt{-1}2mj}{N}\pi) \sum_{m \geq 0} d_m x^m \exp(\frac{-\sqrt{-1}2mj}{N}\pi) \\ &= \sum_{j=1}^N \sum_{m \geq 0} x^m \sum_{k+l=m} d_k d_l \exp(\frac{\sqrt{-1}(k-l)(2j-1)}{N}\pi) \\ &- \sum_{j=1}^N \sum_{m \geq 0} x^m \sum_{k+l=m} d_k d_l \exp(\frac{\sqrt{-1}2(k-l)j}{N}\pi) \\ &= N \sum_{m \geq 0} x^m \sum_{k+l=m, k \equiv l \pmod{N}} d_k d_l \cos(\frac{k-l}{N}\pi) - N \sum_{m \geq 0} x^m \sum_{k+l=m, k \equiv l \pmod{N}} d_k d_l \\ &:= \sum_{m \geq 0} b_m x^m \end{aligned} \quad (47)$$

It's easy to know that all of the coefficients  $b_m$  are nonpositive and infinitely many are negative, then  $h(x)$  and  $\frac{dh(x)}{dx}$  are both negative in  $(0, 1)$ . Hence  $f(x) > 0$  for  $x \in (0, 1)$  by (45).  $\square$

Proof of Theorem 1.8:

*Proof.* By (42), (43) and lemma(3.2), we know (41), (42) and (43) are unsolvable, i.e., if  $L = 2N$ ,  $P_1, P_2, \dots, P_N; \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L$  can not form a central configuration.  $\square$

## References

- [1] Alain Albouy and Vadim Kaloshin. Finiteness of central configurations of five bodies in the plane. *Ann. Math.*, 176(1):535–588, 2012.



- [2] Marshall Hampton and Richard Moeckel. Finiteness of relative equilibria of the four-body problem. *Inventiones Mathematicae*, 163(2):289–312, 2006.
- [3] R. Moeckel and C. Sim. Bifurcation of spatial central configurations from planar ones. *SIAM Journal on Mathematical Analysis*, 26(4):978–998, 1995.
- [4] Richard Moeckel. On central configurations. *Mathematische Zeitschrift*, 205(1):499–517, 1990.
- [5] Julian I Palmore. Classifying relative equilibria. ii. *Bulletin of the American Mathematical Society*, 81(2):489–491, 1975.
- [6] Donald G Saari. On the role and the properties of body central configurations. *Celestial mechanics*, 21(1):9–20, 1980.
- [7] Steve Smale. Mathematical problems for the next century. *The Mathematical Intelligencer*, 20(2):7–15, 1998.
- [8] Aurel Wintner. The analytical foundations of celestial mechanics. *Princeton, NJ, Princeton university press; London, H. Milford, Oxford university press, 1941.*, 1, 1941.
- [9] Xiang Yu and Shiqing Zhang. Twisted angles for central configurations formed by two twisted regular polygons. *Journal of Differential Equations*, 253(7):2106 – 2122, 2012.
- [10] S. Q. Zhang and Q. Zhou. Periodic solutions for planar 2N-body problems. *Proceedings of the American Mathematical Society*, 131:2161–2170, 2003.